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A BIWEIGHT APPROACH TO THE ONE-SAMPLE PROBLEM.(U)

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20. ABSTRACT CONTINUED

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A BIWEIGHT APPROACH TO THE ONE-SAMPLE PROBLEM

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ABSTRACT

A "t"-like statistic, replacing the classical mean by a *biweight in the numerator and the sample* variance by a corresponding variance term in the denominator that modifies that used by Gross ([8]), is proposed and evaluated for its efficiency in constructing confidence intervals in stretched-tailed situations. For the case of many samples of common population width, the method of borrowing estimates of the variance of the numerator via root-mean-squares is considered. The one-sample biweight-"t" is shown, via Monte Carlo simulations, to be very efficient for samples of moderate sizes (in terms of expected length of the confidence interval), and combining variance estimates in the denominator is useful when the underlying distribution of the sample is not extremely long-tailed.

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1. Introduction.

The practice of constructing confidence intervals and making inferences about the location of a single sample using Student's-t is well known. However, it is also well known that this procedure is extremely conservative when the underlying distribution has heavier tails than the Gaussian, for which Student's-t is optimal (see, e.g., [2], [6], [9], [11], [14], [15], [19]). The present study proposes a robust-resistant one-sample statistic which maintains high efficiency, not only if the underlying distribution of the sample is Gaussian, but also if it has somewhat heavier tails.

2. Form of biweight-"t".

The usual t-statistic can be described in the form

$$\frac{\text{estimate of location} - \text{contemplated value}}{\sqrt{\text{estimated variance of the numerator}}}$$

The use of a denominator that "matches" the numerator is the criterion on which, following [19], we focus. The family of M-estimates (e.g., see [10], for a review) immediately suggests itself, since an M-estimate has an asymptotically Gaussian distribution with an explicit variance. One proposal for an M-estimate to be used for this purpose is known as the biweight. The biweight has shown a great deal of promise in recent studies (e.g., [4], [5], [8]).

For a definition of the biweight and its associated

variance, the reader is referred to [13]; we mention here only the computational methods. The biweight estimate of location, T_{bi}, is defined as the solution to the equation

$$\sum_{i=1}^{n} \psi((x_i - T_{bi})/(cs)) = 0 , \tag{1}$$

where

$$\psi(u) = \begin{cases} u(1-u^2)^2 = u \cdot w(u) , & |u| \leq 1 \\ 0 & \text{else} \end{cases}$$

Here, s is an estimate of scale from the sample $x_1,\ldots,x_n$, and c is a multiple of the scale. (A choice of c recommended in [13] is that for which the denominator, $c \cdot s$ is between 4σ and 6σ in the Gaussian case. In this study we will choose c such that $c \cdot s$ is roughly 6σ for the Gaussian.)

We may rewrite (1) in terms of the "weight function", $w(u)$, where

$$w(u) = \psi(u)/u ,$$

whence

$$T_{bi} = \frac{\sum_{i=1}^{n} x_i w(u_i)}{\sum_{i=1}^{n} w(u_i)} , \quad u_i = \frac{x_i - T_{bi}}{c \cdot s} \tag{2}$$

Equation (2) permits an iterative solution. We start the iteration with a robust estimate of location (in this study,

the median of the sample). The location estimate at the kth iteration, $T_{bi}^{(k)}$, $k \geq 1$, is found by

$$T_{bi}^{(k)} = \frac{\sum_{i=1}^{n} x_i w((x_i - T_{bi}^{(k-1)})/(c \cdot s))}{\sum_{j=1}^{n} w((x_i - T_{bi}^{(k-1)})/(c \cdot s))} \tag{3}$$

In determining an estimate of scale to use in (3), former studies (see, e.g., [1], [8]) suggest the median absolute deviation from the median (MAD):

$$s^{(0)} = \text{med} \mid x_i - T_{bi}^{(0)} \mid .$$

For reasons to become clear later, Lax [12] showed that a more efficient scale estimate may be that using the functional form

$$s_{bi} = n^{1/2} \cdot (c_g s^{(0)}) \cdot q_{411}(\{U_i\}) \tag{4}$$

where

$$U_i = \frac{x_i - T^{(0)}}{c_g \cdot s^{(0)}}$$

and

$$q_{411}(\{U_i\}) = \frac{\sum_{i=1}^{n} \psi^2(U_i)}{[\sum_{i=1}^{n} \psi'(U_i)][\max(1,-1+\sum_{i=1}^{n} \psi'(U_i))]} \tag{5}$$

Here, as before, $T^{(0)}$ is the median of the sample, $s^{(0)}$

is the MAD, and c_8 is again chosen in order that $c_8 \cdot s^{(8)}$ is approximately the desired multiple of σ in the Gaussian case. (Since $s^{(8)} \doteq (2/3)\sigma$ for a Gaussian sample, we choose $c_8 = 9$ for this calculation.)

Finally, the denominator of our "t"$_{bi}$ statistic is given by S_{bi}, where S_{bi}^2 estimates the variance of T_{bi}. Huber [18] derives the theoretical asymptotic variance of T_{bi}, from which we may obtain a finite-sample approximation to it as

$$S_{bi}^2 = \hat{\text{Var}}(T_{bi}) = (c \cdot s_{bi})^2 q(\{u_i\}), \qquad (6)$$

where

$$u_i = \frac{x_i - T_{bi}}{c \cdot s_{bi}},$$

as in equation (4). Notice that, <u>in functional form</u>,

$$S_{bi}^2 = s_{bi}^2/n,$$

just as

$$\hat{\text{Var}}(\bar{X}) = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n(n-1)} = \frac{\text{classical sample } s^2}{n}$$

in the Gaussian case. However, s_{bi}^2 uses the median and the MAD in its computation, whereas S_{bi}^2 uses the more advanced location and scale estimates T_{bi} and s_{bi}. Notice also that $q_{411}(\{u_i\})$ as defined in (5) may be written

$$q_{411}(\{u_i\}) = \frac{\sum_{i=1}^{n} u_i^2 (1-u_i^2)^4}{\left[\sum_{i=1}^{n} (1-u_i^2)(1-5u_i^2)\right]\left[\max\left(1, -1 + \sum_{i=1}^{n} (1-u_i^2)(1-5u_i^2)\right)\right]}$$

The exponents of $(1-u_i^2)$, $(1-u_i^2)$, and $(1-5u_i^2)$, respectively, suggest the subscript and the name "411 widther" for s_{bi} (Equation (4)). Our biweight-"t" statistic then takes the form

$$\text{"t"}_{bi} = \frac{T_{bi}}{S_{bi}}.$$

3. Methods of evaluation.

(a) Situations.

Since "Real data. . . often has stretched-out tails," ([13], p.23), we shall be chiefly interested in high performance of our robust-resistant-"t", not only in the (unlikely) event that our data does in fact come from a Gaussian distribution, but also in the (more common) situation that the underlying distribution has somewhat heavier tails. To this end, we shall consider performance in three underlying situations (each unlikely in a different way):

- o Gaussian
- o "One-Wild", or a contaminated Gaussian, in which exactly one (unspecified) observaton comes from a Gaussian distribution with 100 times the variance
- o "Slash" = Gaussian deviate/independent uniform,

an extremely stretched-tailed distribution, "as unrealistic as the Gaussian but in the opposite direction" ([17]).

These three sampling situations are likely to cover a reasonably broad range of tail behavior. (See [16] for a further description of the stretched-tailed distributions.) The term "robustness of efficiency" is often used to refer to relatively good performance over a range of distributions.

(b) Efficiency of confidence intervals.

One measure of reasonably good confidence intervals is their average length (hopefully short). The expected confidence interval length (ECIL) of probability $(1 - \alpha)$ was defined and used by Gross [8] as

ECIL = $2(\alpha$-% point of tabular t)·ave(denominator of "t")

Since we will be considering more than one error rate, we will emphasize the dependence of this quantity on α. Furthermore, we will distinguish between "exact" ECIL

$$\mathrm{ECIL}(\alpha) = 2 \times (\alpha\text{-\% point of biweight-"t"}) \times \mathrm{ave}(\text{denominator of } \text{"t"}_{bi}) \tag{7}$$

and an "approximate" ECIL

$$\mathrm{ECIL}(\alpha,\nu) = 2\ (\alpha\text{-\% point of Student's-}t_{\nu}) \times \mathrm{ave}(\text{denominator of } \text{"t"}_{bi}) \tag{8}$$

We shall consider (8) when we approximate the distribution of "t"$_{bi}$ by one from the Student's-t family. From (7) and (8), we may define <u>ECIL</u> <u>efficiency</u> as:

$$\mathrm{eff}(\alpha) = \left[\frac{\mathrm{ECIL}_D(\alpha)}{\mathrm{ECIL}(\alpha)}\right]^2 \tag{9}$$

and

$$\mathrm{eff}(\alpha,\nu) = \left[\frac{\mathrm{ECIL}_D(\alpha)}{\mathrm{ECIL}(\alpha,\nu)}\right]^2 \tag{10}$$

where

$\mathrm{ECIL}_D(\alpha)$ = shortest ECIL for situation D.

where the "shortest" is over all possible confidence intervals. (Except for the Gaussian case, we shall have to use an approximation to this.) Notice that the quantities in (9) and (10) are squared to set them on a variance scale. For the three distributional situations considered in this study, we may determine $\mathrm{ECIL}_D(\alpha)$ exactly from the Student procedure in the Gaussian case (n observations) or approximately for the One-Wild (n-1 "good" observations), and approximately from the Slash density (see Appendix 1).

(c) Approximating degrees of freedom.

In order to use biweight-"t" to construct confidence intervals, we need to know the critical points of its distribution. Practically speaking, we prefer to approximate this distribution by something fairly simple. The most promising candidate here is one from the Student's-t family. Given a tail area, α, and having computed the associated critical point, "t"$_{bi}(\alpha)$ (via Monte Carlo: see Section 4),

we interpolate on the reciprocal of the degrees of freedom from standard Student's-t tables to obtain the closest degrees of freedom (to one decimal place) to which our biweight-"t" corresponds. Gross ([8]) suggested that using a denominator of the form

$$\frac{\sum_{i=1}^{n} \psi^2(u)}{\left(\sum_{i=1}^{n} \psi'(u)\right)^2},$$

roughly half the nominal degrees of freedom gave conservative approximations at the 5%-point. Notice that our denominator in (6) is a modified version of that used by Gross.

4. Calculations.

Percent points of biweight-"t" were calculated via a Monte Carlo swindle to achieve relatively high accuracy with only a moderate number of samples. For all situations except One-Wild (n=5) and Slash (n=5), these samples were those used in the Princeton Robustness Study ([1], Ch. 12). There are typically 640 or 1000 samples in each distribution situation of a given sample size. Details concerning the theory of the specific Monte Carlo technique may also be found in [1] (Ch. 4 and Appendix 3) or in [18]. The particular application of the technique followed the procedure outlined in [7].

For each run of a "t" statistic formed as a biweight

numerator and trial denominator, we evaluated values of ECIL efficiency (9) for different levels of α, where α is the nominal level. To compute the biweight, the w-iteration of Equation (3) was terminated at the kth iteration when either $|T_k - T_{k-1}| \le .0005$ or $k \ge 15$. For most samples in this study (for which the number of observations ranged from 5† to 20), the first convergence criterion was typically satisfied within 3-5 iterations. For the Slash distribution, the number of iterations was often slightly higher (5-8); in roughly 5% of the cases, the iteration was terminated by virtue of the limit of 15. Over all distributions, however, fewer than 10 iterations was required in more than 90% of the samples.

Denoting the α-percent point of the biweight-"t" distribution by

$$\text{"}t\text{"}(\alpha)$$

and that from the Student's-t distribution on ν degrees of freedom by

$$t_\nu(\alpha)$$

the tables also present the values of

$$r_1(\alpha) = \log_{10}(\text{"}t\text{"}(\alpha)/t_{.9(ndf)}(\alpha))$$

and

†Results on sample size $n = 5$ are reported separately in "A robust confidence interval for a sample of five observations," Tech. Report No. 152, Series 2, Department of Statistics, Princeton University.

$$r_2(d) = 100 \cdot P\{\text{biweight-"t"} \geq t_{.9(ndf)}(d)\}/d \qquad (11)$$

where

ndf = nominal degrees of freedom.

These two columns provide us with an indication of the success in approximating the biweight-"t" distribution from one from the Student's-t family on 0.9(nominal degrees of freedom). Notice that, ideally,

$$r_1(d) = 0 \text{ (possibly negative)}$$

$$r_2(d) = 100\% \text{ (possibly less)}$$

A more precise matching on degrees of freedom is given by the sixth column, in which the correspondence

(critical point, d) --> df

is made.

5. Results for n=10 and n=20.

As shown in Exhibit 1, the overall efficiency of biweight-"t" confidence intervals, as measured in terms of Equation (9), nearly exceeds 75% in all situations. In fact, this efficiency even rises in the Gaussian and One-Wild situations for the more extreme percentage points. At n=10, there is a slight loss of efficiency in the extreme tails of one stretched-tailed situation, but the loss is relatively small even at the .01%-point. The matching of tail area and critical points to a Student's-t distribution suggests that we may conservatively approximate the distribution of biweight-"t" using 0.75x(nominal df) for n=10.

The corresponding results for n=20 are shown in Exhibit 2. Here, efficiencies exceed 90% for the Gaussian and One-Wild (a relatively mild departure), and rarely drop below 75% for the Slash. We can match Student's-t on 0.9x(nominal df) conservatively, and still have highly efficient confidence intervals.

For comparison, Exhibits 3 and 4 provide the corresponding results on the stretched-tailed distributions for the classical Student's-t using the mean, for which the confidence intervals become horribly long, even in the case of low percentage contamination (5% for One-Wild at n=20). (For n=10, the relative efficiency of the biweight is 80%, 310%, and 200% of that of Student's-t for the Gaussian, One-Wild, and Slash situations; at n=20, the relative efficiencies are roughly 90%, 310%, and 5000%.)

6. Borrowing denominators.

If we have, say, J samples, all believed to have common width, we might consider combining estimates of the variance of the numerator via root-mean-squares:

$$S = \sqrt{(S_1^2+\ldots+S_J^2)/J}\ , \quad J=2,3,4,5 \qquad (12)$$

Such a borrowed denominator would be common in an analysis of variance format comparing J treatments of n observations per treatment, for which the usual pooled mean squared error term in a truly Gaussian situation is distributed as a multiple of $\chi^2_{J(n-1)}$. (One could also borrow by taking a

median, or a midmean, of S_j^2, rather than an average, in Equation (12), but we have not yet investigated this possibility.)

Exhibits 5 and 6 show the results of borrowing J denominators in our "t" statistic. In a Gaussian situation, S_{bi}^2 does not appear to be behaving too much differently from the typical S^2_{sample}; we find a systematic increase in the degrees of freedom and little change in the ECIL efficiencies. The distribution of biweight-"t" when the underlying samples all contain one wild observation becomes increasingly shorter-tailed as the denominator pools from more samples; thus Jx(approximate degrees of freedom from one-sample results) is a conservative matching here.

The borrowing for Slash is less successful, for here we are entitled to an increase up to only J=2; thereafter, borrowing is not necessarily profitable. Returning to the assumptions underlying the use of (12), we recall that this form would be used only when we believed our samples came from populations with common width. While this is certainly true for all the samples in the simulation of the Slash distribution, highly different values of S are more likely to occur with this situation, simply because of its extremely heavy tails. For example, the first 15 values of S in the simulation for n=10 are:

0.4420, 1.0352, 0.9646, 1.0363, 0.5038,
1.2933, 0.7951, 2.4666, 1.0426, 1.1087,

1.5627, 1.2563, 1.2340, 2.0614, 0.6349

An unusual value of the denominator contributes to J of the borrowed denominators in the simulation. The increased variability in the denominators which are borrowed is likely to affect the success of the overall procedure, thus suggesting that other forms of combination should be investigated.

7. Observations and Discussion.

The previous results indicate that we can be very sure about the efficiency of our biweight-"t" confidence intervals in the case of one sample, where $n \geq 10$. In fact,

(1) Efficiencies usually run 75% , and often as high as 90%. In practical terms, this means that our biweight-"t" intervals are running only slightly longer than the best we could possibly do if we knew the situation at hand, which of course we never do.

(2) For testing purposes, we may conservatively approximate the distribution of biweight-"t" by a Student's-t on

$$0.75(n-1) \text{ df for } 10 \leq n \leq 20$$
$$0.9(n-1) \text{ df for } n \geq 20$$

and be confident of the conservatism of our approximation even at extreme percentage points.

Exhibit 7 plots the approximate efficiency (Eqn.10) in

the expected length of the confidence intervals when we compare biweight-"t" to a Student's-t on 0.75x(nominal df) for n=10; i.e.,

$$\mathrm{eff}(\alpha,7) = [\mathrm{ECIL}_D(\alpha)/\mathrm{ECIL}(\alpha,7)]^2$$

In Exhibit 8, we plot eff(α,.9(ndf))=eff(α,17) for samples of size 20. Notice that, on the average, the efficiency is often above 65% when we construct our confidence intervals in this way. These plots are to be compared with those in Exhibits 9 and 10, which show the expected efficiency when the usual Student's-t procedure on the nominal (n-1) degrees of freedom is used.

(3) If we believe we have sampled two populations of common scale, we may construct our denominator by borrowing from both samples. In most practical situations, borrowing over more than two samples is likely to be desirable, and the use of appropriately increased numbers of degrees of freedom is reasonable. In the case of extremely stretched-tailed situations, further borrowing may be over-optimistic.

Appendix: A "t" statistic tailored to the Slash density.

The Slash (ratio of independent Gaussian and Uniform(0,1) random variables) density is given by:

$$\frac{1}{\sigma}f((x-\mu)/\sigma) = \begin{cases} [1-\exp(-1/2((x-\mu)/\sigma)^2)]/[\sqrt{2\pi}\,\sigma((x-\mu)/\sigma)^2] & x\neq\mu \\ 1/(2\sigma\sqrt{2\pi}) & x=\mu \end{cases}$$

Letting $x=((y-\mu)/\sigma)$, the ML equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial\mu}\ln[\frac{1}{\sigma}f(x_i)] = 0$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial\sigma}\ln[\frac{1}{\sigma}f(x_i)] = 0$$

become

$$\sum_{i=1}^{n} -f'(x_i)/f(x_i) = \sum_{i=1}^{n} x_i w(x_i) = 0$$

$$\sum_{i=1}^{n} \left(-\frac{1}{\sigma} - \left(\frac{y-\mu}{\sigma^2}\right)\frac{f'(x_i)}{f(x_i)} = \frac{1}{\sigma}\sum_{i=1}^{n} (-1-x_i\frac{f'(x_i)}{f(x_i)}\right)$$

$$= \frac{1}{\sigma}\sum_{i=1}^{n} (-1+x_i^2 w(x_i)) = 0$$

where

$$w(x_i) = -\frac{f'(x_i)}{x_i f(x_i)}$$

The first of these leads to the usual w-iteration (as in Eqn. 3 of Sec.2) for location estimates, namely

$$0 = \sum_{i=1}^{n} ((y_i-\mu)/\sigma)w(x_i) \Longrightarrow \hat{\mu}_{s1} = \frac{\sum_{i=1}^{n} y_i w(x_i)}{\sum_{i=1}^{n} w(x_i)} . \qquad (A1)$$

The second can be written to solve iteratively for σ as

$$1 = (1/n)\cdot \sum_{i=1}^{n} x_i^2 w(x_i)$$

$$\Longrightarrow \sigma^2 = (\sigma^2/n) \sum_{i=1}^{n} x_i^2 w(x_i)$$

In the case of the Slash density,

$$w(x) = -\frac{f'(x)}{xf(x)} = \frac{2}{x^2}\left|1 - \frac{x^2/2}{e^{x^2/2}-1}\right|$$

For large x, clearly the first term dominates, so

$$w(x) \simeq 2/x^2.$$

(For $x>7$, the error in approximation is less than 3×10^{-11}.)

For small x, we expand $(e^{x^2/2} - 1)$ as a Taylor series, for which the dominant term is

$$w(x) = \frac{x^2}{4+x^2} + O(1/x^4).$$

The asymptotic variance of $n^{1/2}\hat{\mu}$ (see Huber, 1964) is

$$\frac{\int \Psi^2(x)\, dF(x)}{\int (\Psi'(x))^2\, dF(x)} \qquad (A2)$$

whence, for the Slash density, we have

$$\Psi(x) = x\cdot w(x) = \frac{2}{x} - \frac{x}{e^{x^2/2}-1}$$

and

$$\Psi'(x) = \frac{-2(e^{x^2/2}-1)^2 + x^4 e^{x^2/2} - x^2(e^{x^2/2}-1)}{x^2(e^{x^2/2}-1)^2}$$

It is easy (though somewhat tedious) to show that both integrals in the numerator and denominator of (A2) are bounded. Thus,

$$\hat{Var}(\hat{\mu}_{s1}) = \hat{\sigma}^2 \cdot \frac{\sum_{i=1}^{n} \Psi^2(x_i)}{[\Sigma\Psi'(x_i)][-\Psi'_{max} + \Sigma\Psi'(x_i)]} .^{\dagger}$$

For large x,

$$\Psi(x) \simeq 2/x$$

$$\Psi'(x) \simeq -2/x^2,$$

and for small x,

$$\Psi(x) \simeq \frac{2(x^2/2+x^4/8+\ldots)-x^2}{(x^3/2+x^5/8+\ldots)} \simeq x/2$$

$$\Psi'(x) \simeq (x^2+4)/(4x^2+8) \longrightarrow 1/2 \text{ as } x \to 0. \qquad (A3)$$

From (A3) we also see that for all $x \neq 0$,

† The quantity $\psi'_{max} = \max_u \psi'(u)$ is one of a number of possible *homogeneous* functions of ψ which may be used. The justification for considering homogeneous functions seems natural in light of dealing with random variables $y_i = \psi(x_i)$; see [10]. Note that for the biweight, $\psi'_{max} = 1$, as in Equation (5).

$$\psi'(x) < 1/2 \Longrightarrow \psi'_{max} = 1/2.$$

The "t" statistic based on the maximum likelihood estimates for the Slash density is then

$$\text{"t"}_{sl} = \frac{\hat{\mu}_{sl}}{S_{sl}} \quad \text{where} \quad S^2_{sl} = \hat{\text{Var}}(\hat{\mu}_{sl})$$

A $100(1-2\alpha)\%$ confidence interval based on this statistic would have expected length

$$2\cdot(\alpha\%\text{-point of "t"}_{sl})\cdot \text{ave}(S_{sl}) \qquad \text{(A4)}$$

<u>A caveat</u>. Unlike the Gaussian situation, where the use of Student's-t provides maximum power and $S^2_{sample} = [\sum_{i=1}^{n} (y_i-\bar{y})^2]/(n(n-1))$ has uniformly minimum variance, we have no corresponding results concerning the use of "t"$_{sl}$ and S_{sl}. Thus, the resulting confidence intervals are not necessarily the shortest available. The use of "t"$_{sl}$ is justified by its analogous form to the classical Student's-t, and by the asymptotic efficiency of $\hat{\mu}$ as a maximum likelihood estimate. Table A1.1 shows the values of (A4) for various α-levels used in this study.

Finally, we can apply the same procedure when we wish to obtain maximum likelihood estimates in the two-sample problem involving the Slash density. For example, if sample $\underline{x}$ ~ Gaussian, and sample $\underline{y}$ ~ Slash, we could approximate the tail distribution of

$$\text{"t"}_{GS} = \frac{\hat{\mu}_{sl}(\underline{X}) - \hat{\mu}_{sl}(\underline{Y})}{\sqrt{S^2_{sample}(\underline{X}) + S^2_{sl}(\underline{Y})}}$$

and use its $\alpha\%$-point and the average value of its denominator in approximating the "shortest" confidence interval length as in (A4).

Table A1.1: Expected lengths of 100(1-2α)% confidence intervals based on ML estimates for Slash density (one-sample)

n = 5

α	J=1	J=2	J=3	J=4	J=5
0.00001	182.3	12.01	7.62	7.93	5.45
0.000025	161.0	11.02	6.86	7.11	4.80
0.00005	158.4	10.29	6.27	6.47	4.24
0.0001	140.8	9.48	6.66	5.80	3.62
0.0005	61.84	7.05	3.98	3.97	2.35
0.001	33.45	5.79	3.11	2.88	1.99
0.005	10.61	3.13	1.79	1.51	1.34
0.01	6.93	2.27	1.45	1.24	1.11
0.025	3.59	1.46	1.07	0.93	0.84
0.05	2.04	1.04	0.82	0.71	0.65

n = 10

α	J=1	J=2	J=3	J=4	J=5
0.00001	12.10	11.18	9.64	9.31	9.11
0.000025	11.26	10.26	8.99	8.69	8.59
0.00005	10.06	9.54	8.48	8.22	8.19
0.0001	9.92	8.81	7.97	7.76	7.77
0.0005	8.29	7.13	6.74	6.64	6.69
0.001	7.57	6.43	6.20	6.12	6.18
0.005	5.81	4.92	4.88	4.86	4.87
0.01	5.00	4.29	4.28	4.27	4.26
0.025	3.895	3.47	3.45	3.45	3.43
0.05	3.06	2.82	2.80	2.79	2.78

n = 20

α	J=1	J=2	J=3	J=4	J=5
0.00001	6.45	6.37	5.22	5.11	5.32
0.000025	5.99	6.00	4.96	4.84	5.05
0.00005	5.63	5.71	4.75	4.64	4.83
0.0001	5.27	5.40	4.53	4.42	4.56
0.0005	4.42	4.56	3.98	4.86	4.00
0.001	4.06	4.16	3.72	3.60	3.68
0.005	3.20	3.21	3.03	2.94	2.96
0.01	2.81	2.81	2.70	2.63	2.63
0.025	2.29	2.28	2.23	2.19	2.18
0.05	1.87	1.87	1.84	1.83	1.82

Note: J=# of borrowed (root-mean-square) denominators.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
Dist'n. Gaussian	0.00001	181.7	0.028	9.505	(.378)	7.5	6.004	68.9
	0.000025	195.7	0.038	8.575	(.353)	7.2	5.416	67.2
	0.00005	188.5	0.041	7.827	(.033)	7.1	4.944	67.3
	0.00010	172.2	0.039	7.053	(.028)	7.0	4.455	68.9
	0.00050	127.7	0.021	5.289	(.014)	7.3	3.341	77.5
	0.00100	112.7	0.010	4.611	(.011)	7.6	2.912	82.4
	0.00500	93.5	-.006	3.308	(.046)	8.4	2.090	91.6
	0.01000	91.1	-.009	2.836	(.032)	8.8	1.791	94.0
	0.02500	91.4	-.011	2.250	(.021)	9.3	1.421	95.9
	0.05000	93.2	-.011	1.815	(.015)	9.9	1.147	96.8
Dist'n. One-Wild	0.00001	584.7	0.105	11.34	(.048)	6.4	8.100	50.5
	0.000025	454.8	0.112	10.15	(.046)	6.1	7.251	49.0
	0.00005	308.1	0.109	9.147	(.045)	6.0	6.532	49.6
	0.00010	277.9	0.097	8.058	(.040)	6.0	5.754	52.3
	0.00050	162.9	0.045	5.629	(.019)	6.6	4.020	65.7
	0.00100	136.2	0.029	4.812	(.013)	6.9	3.440	71.5
	0.00500	103.5	0.003	2.382	(.050)	7.8	2.415	80.6
	0.01000	97.5	-.003	2.879	(.034)	8.2	2.036	82.9
	0.02500	94.5	-.007	2.270	(.026)	8.8	1.621	84.5
	0.05000	95.1	-.008	1.821	(.016)	9.2	1.305	84.8
Dist'n. Slash	0.00001	74.0	-.014	8.623	(.334)	8.4	16.09	56.6
	0.000025	89.4	-.006	7.742	(.300)	8.2	14.45	60.7
	0.00005	94.1	-.004	7.060	(.267)	8.1	13.17	58.3
	0.00010	92.8	-.005	6.367	(.229)	8.2	11.88	69.7
	0.00050	76.5	-.022	4.794	(.142)	8.9	8.944	85.9
	0.00100	69.8	-.030	4.199	(.097)	9.6	7.832	93.3
	0.00500	65.9	-.034	3.103	(.051)	11.0	5.791	100.7
	0.01000	71.5	-.029	2.708	(.038)	11.3	5.053	96.8
	0.02500	83.0	-.019	2.205	(.031)	10.8	4.115	89.6
	0.05000	93.6	-.009	1.823	(.026)	9.5	3.401	81.0

Exhibit 1. Results on one-sample biweight - "t", n = 10, scaled using ASYMV widther.
(a): Fixed scale.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	58.6	-.017	5.602	(.068)	19.3	2.524	96.8
	0.000025	67.1	-.014	5.205	(.058)	19.0	2.345	96.1
	0.00005	72.5	-.012	4.903	(.051)	18.9	2.209	95.7
Dist'n.	0.00010	77.0	-.011	4.600	(.042)	18.8	2.072	95.6
Gaussian	0.00050	84.6	-.008	3.892	(.031)	18.8	1.753	95.6
	0.00100	87.0	-.007	3.584	(.025)	18.8	1.615	95.7
	0.00500	91.4	-.006	2.858	(.018)	19.2	1.287	96.2
	0.01000	93.0	-.006	2.533	(.015)	19.6	1.141	96.5
	0.02500	95.0	-.025	2.085	(.012)	20.2	0.939	96.8
	0.05000	96.5	-.005	1.720	(.009)	21.1	0.775	97.0
	0.00001	49.2	-.022	5.541	(.067)	20.1	2.628	97.1
	0.000025	59.1	-.018	5.158	(.061)	19.8	2.446	95.7
	0.00005	65.4	-.016	4.862	(.055)	19.6	2.306	95.0
Dist'n.	0.00010	70.4	-.014	4.562	(.047)	19.6	2.164	94.5
One-Wild	0.00050	78.0	-.012	3.856	(.037)	19.9	1.829	94.2
	0.00100	80.2	-.012	3.548	(.031)	20.2	1.683	94.3
	0.00500	84.8	-.011	2.824	(.022)	21.7	1.339	94.6
	0.01000	87.1	-.011	2.503	(.018)	22.7	1.187	94.7
	0.02500	90.3	-.010	2.060	(.014)	25.0	0.977	94.8
	0.05000	93.0	-.010	1.701	(.011)	28.0	0.807	94.6
	0.00001	97.9	-.001	5.821	(.164)	17.1	7.470	74.6
	0.000025	92.4	-.003	5.337	(.140)	17.5	6.850	76.5
	0.00005	87.5	-.006	4.975	(.121)	17.9	6.384	77.8
Dist'n.	0.00010	82.9	-.009	4.620	(.099)	18.5	5.929	79.1
Slash	0.00050	76.6	-.014	3.841	(.069)	20.4	4.929	80.6
	0.00100	76.2	-.015	3.523	(.056)	21.3	4.521	80.6
	0.00500	82.1	-.013	2.812	(.043)	22.6	3.609	78.4
	0.01000	86.8	-.011	2.523	(.038)	22.6	3.213	76.7
	0.02500	92.8	-.007	2.074	(.032)	21.9	2.662	73.9
	0.50000	97.1	-.004	1.725	(.026)	20.0	2.213	71.3

Exhibit 2. Results on one-sample biweight - "t", n = 20, scaled using fixed ASYMV widther.

	Tail Pr.	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	7.532	(.134)	10.0	4.633	187.6
	0.000025	6.893	(.118)	9.7	4.240	104.7
	0.00005	6.401	(.106)	9.5	3.938	103.0
Dist'n.	0.00010	5.904	(.082)	9.3	3.632	101.8
Gaussian	0.00050	4.759	(.064)	9.1	2.927	100.5
	0.00100	4.283	(.049)	9.1	2.635	100.3
	0.00500	3.241	(.031)	9.1	1.994	100.3
	0.01000	2.815	(.024)	9.1	1.732	100.2
	0.02500	2.260	(.018)	9.1	1.390	100.1
	0.05000	1.833	(.014)	9.0	1.127	100.0
	0.00001	7.077	(.270)	11.1	12.715	20.5
	0.000025	6.222	(.253)	11.7	11.179	20.5
	0.00005	5.576	(.214)	12.5	10.019	21.1
Dist'n.	0.00010	4.982	(.167)	14.0	8.951	21.6
One-Wild	0.00050	3.804	(.082)	21.5	6.836	22.7
	0.00100	3.378	(.057)	30.7	6.069	23.0
	0.00500	2.510	(.026)	∞	4.510	23.1
	0.01000	2.173	(.020)	∞	3.905	23.0
	0.02500	1.763	(.016)	∞	3.168	22.1
	0.05000	1.487	(.012)	∞	2.673	20.2
	0.00001	5.351	(.114)	23.0	58.65	4.3
	0.000025	4.935	(.106)	23.9	54.08	4.3
	0.00005	4.619	(.094)	25.1	50.63	3.9
Dist'n.	0.00010	4.307	(.071)	26.8	47.21	4.4
Slash	0.00050	3.600	(.053)	34.2	39.46	4.4
	0.00100	3.303	(.051)	41.0	36.20	4.4
	0.00500	2.617	(.038)	123.0	28.68	4.1
	0.01000	2.321	(.037)	∞	25.44	3.9
	0.02500	1.927	(.032)	∞	21.12	3.4
	0.05000	1.639	(.024)	∞	17.96	2.9

<u>Exhibit 3.</u> Results of classical Student's - t (9 d.f.), n = 10.

	Tail Pr.	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	5.495	(.054)	20.7	2.425	104.9
	0.000025	5.121	(.049)	20.2		
	0.00005	4.838	(.040)	20.0	2.136	102.5
Dist'n.	0.00010	4.551	(.037)	19.7	2.008	101.7
Gaussian	0.00050	3.870	(.027)	19.4	1.708	100.7
	0.00100	3.572	(.023)	19.2	1.576	100.4
	0.00500	2.861	(.016)	19.0	1.263	100.1
	0.01000	2.541	(.012)	18.8	1.122	99.9
	0.02500	2.096	(.010)	18.6	0.925	99.7
	0.05000	1.733	(.008)	18.2	0.765	99.4
	0.00001	4.815	(.074)	41.6	4.727	30.0
	0.000025	4.466	(.058)	47.3	4.385	29.8
	0.00005	4.198	(.053)	55.1	4.122	29.7
Dist'n.	0.00010	3.928	(.051)	70.2	3.857	29.7
One-Wild	0.00050	3.292	(.032)	∞	3.232	30.1
	0.00100	3.016	(.031)	∞	2.962	30.4
	0.00500	2.380	(.021)	∞	2.337	31.1
	0.01000	2.111	(.019)	∞	2.072	31.1
	0.02500	1.763	(.014)	∞	1.732	30.2
	0.05000	1.519	(.009)	∞	1.492	27.7
	0.00001	4.519	(.119)	86.1	47.449	1.85
	0.000025	4.180	(.116)	150.3	43.882	1.86
	0.00005	3.921	(.114)	379.3	41.164	1.87
Dist'n.	0.00010	3.666	(.105)	∞	38.491	1.88
Slash	0.00050	3.129	(.079)	∞	32.857	1.81
	0.00100	2.910	(.082)	∞	30.548	1.76
	0.00500	2.391	(.058)	∞	25.099	1.62
	0.01000	2.156	(.054)	∞	22.639	1.54
	0.02500	1.804	(.042)	∞	18.946	1.46
	0.05000	1.536	(.030)	∞	16.122	1.34

Exhibit 4. Results of classical Student's - t (19 d.f.), n = 20.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	114.0	0.006	6.039	(.199)	15.5	3.871	85.0
	0.000025	101.0	0.0005	5.490	(.168)	16.0	3.519	87.8
	0.00005	92.6	-.004	5.091	(.136)	16.5	3.264	90.1
Dist'n. Gaussian	0.00010	86.4	-.007	4.713	(.102)	17.0	3.021	92.1
	0.00050	79.9	-.012	3.908	(.053)	18.4	2.505	95.4
	0.00100	79.9	-.013	3.582	(.038)	18.9	2.296	96.2
	0.00500	83.6	-.013	2.838	(.021)	20.6	1.819	97.4
	0.01000	86.1	-.012	2.513	(.016)	21.5	1.611	97.7
	0.02500	89.7	-.011	2.066	(.012)	23.5	1.325	97.9
	0.05000	92.6	-.010	1.705	(.009)	26.4	1.093	98.0
	0.00001	69.7	-.013	5.781	(.107)	17.5	4.226	85.7
	0.000025	73.3	-.012	5.332	(.088)	17.5	3.898	85.2
	0.00005	75.4	-.012	4.998	(.074)	17.6	3.654	85.0
Dist'n. One-Wild	0.00010	77.1	-.011	4.668	(.056)	17.7	3.412	84.9
	0.00050	79.9	-.012	3.910	(.040)	18.3	2.859	85.0
	0.00100	80.8	-.012	3.587	(.032)	18.7	2.622	85.1
	0.00500	83.6	-.013	2.837	(.021)	20.6	2.074	85.5
	0.01000	85.6	-.013	2.510	(.017)	21.9	1.835	85.4
	0.02500	89.2	-.012	2.063	(.013)	24.2	1.508	83.1
	0.05000	92.4	-.011	1.704	(.010)	26.7	1.246	84.7
	0.00001	7.9	-.069	5.089	(.082)	29.1	9.972	125.7
	0.000025	12.5	-.059	4.787	(.075)	28.1	9.379	119.7
	0.00005	17.5	-.051	4.561	(.076)	27.0	8.936	114.0
Dist'n. Slash	0.00010	24.7	-.044	4.333	(.078)	25.8	8.490	107.7
	0.00050	67.0	-.027	3.774	(.082)	22.8	7.396	92.9
	0.00100	63.4	-.021	3.513	(.067)	21.7	6.884	87.2
	0.00500	86.6	-.009	2.861	(.055)	19.6	5.606	77.0
	0.01000	94.9	-.004	2.561	(.046)	17.4	5.018	73.1
	0.02500	102.4	0.002	2.132	(.038)	14.9	4.178	69.0
	0.05000	103.0	0.004	1.763	(.032)	13.8	3.455	66.6

Exhibit 5 (a). Results on one-sample biweight - "t", n = 10, borrowing denominators from 2 samples.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	65.5	-.014	5.120	(.087)	28.1	3.298	96.0
	0.000025	64.9	-.015	4.757	(.072)	29.1	3.064	97.0
	0.00005	65.1	-.016	4.488	(.060)	29.9	2.891	97.6
Dist'n.	0.00010	65.9	-.016	4.223	(.045)	30.8	2.720	98.1
Gaussian	0.00050	70.6	-.015	3.619	(.029)	32.9	2.328	98.7
	0.00100	73.6	-.015	3.350	(.022)	33.9	2.158	98.7
	0.00500	81.6	-.013	2.715	(.014)	37.4	1.748	98.6
	0.01000	85.1	-.012	2.424	(.012)	40.0	1.561	98.5
	0.02500	89.5	-.011	2.013	(.009)	46.4	1.296	98.4
	0.05000	92.5	-.010	1.671	(.007)	58.3	1.077	98.3
	0.00001	49.8	-.021	5.039	(.060)	30.7	3.713	88.3
	0.000025	54.4	-.020	4.706	(.053)	31.1	3.468	87.9
	0.00005	57.5	-.019	4.452	(.046)	31.6	3.281	87.6
Dist'n.	0.00010	60.4	-.019	4.196	(.036)	32.3	3.019	87.4
One-Wild	0.00050	67.0	-.018	3.595	(.027)	34.8	2.649	87.0
	0.00100	70.0	-.017	3.331	(.021)	36.4	2.455	86.8
	0.00500	78.1	-.016	2.697	(.015)	42.5	1.988	85.7
	0.01000	82.0	-.015	2.409	(.013)	47.0	1.775	85.8
	0.02500	87.4	-.013	2.002	(.010)	57.7	1.475	85.2
	0.05000	91.3	-.012	1.665	(.008)	76.6	1.227	84.7
	0.00001	133.3	0.006	5.367	(.193)	22.7	10.71	81.0
	0.000025	181.0	0.016	5.110	(.202)	20.5	10.20	77.7
	0.00005	201.5	0.022	4.895	(.208)	19.0	9.770	75.3
Dist'n.	0.00010	204.9	0.026	4.657	(.211)	17.9	9.295	73.5
Slash	0.00050	172.8	0.028	3.996	(.160)	16.4	7.977	71.4
	0.00100	158.3	0.026	3.679	(.144)	16.2	7.343	71.3
	0.00500	136.0	0.022	2.940	(.069)	15.2	5.869	69.1
	0.01000	128.1	0.021	2.614	(.058)	14.5	5.218	67.3
	0.02500	115.8	0.017	2.145	(.044)	14.0	4.281	64.9
	0.05000	106.1	0.009	1.747	(.035)	15.8	3.487	64.5

<u>Exhibit 5 (b)</u>. Results on one-sample biweight - "t", n = 10, borrowing denominators from 3 samples.

Dist'n.	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	60.1	-.015	4.837	(.053)	40.5	3.119	97.6
	0.000025	63.0	-.014	4.529	(.046)	41.4	2.924	87.8
	0.00005	65.3	-.014	4.299	(.039)	42.2	2.775	98.0
Gaussian	0.00010	67.7	-.014	4.067	(.031)	43.1	2.626	98.1
	0.00050	73.9	-.013	3.517	(.023)	46.0	2.270	98.2
	0.00100	76.8	-.012	3.272	(.018)	47.7	2.112	98.2
	0.00500	83.8	-.011	2.670	(.012)	54.4	1.724	98.2
	0.01000	86.7	-.011	2.390	(.010)	60.1	1.543	98.2
	0.02500	90.4	-.010	1.992	(.008)	76.3	1.286	98.2
	0.05000	93.1	-.009	1.658	(.006)	115.8	1.070	98.0
	0.00001	48.2	-.020	4.768	(.049)	45.2	3.527	87.9
	0.000025	52.5	-.020	4.475	(.043)	46.3	3.310	87.5
	0.00005	55.6	-.019	4.250	(.038)	47.5	3.144	87.3
One-Wild	0.00010	58.6	-.019	4.022	(.036)	49.2	2.975	87.1
	0.00050	65.7	-.018	3.478	(.023)	55.0	2.573	86.7
	0.00100	69.1	-.017	3.236	(.018)	58.9	2.393	86.5
	0.00500	77.8	-.015	2.643	(.013)	75.7	1.957	85.9
	0.01000	81.9	-.015	2.368	(.011)	91.2	1.752	85.5
	0.02500	87.5	-.013	1.976	(.009)	146.1	1.462	85.0
	0.05000	91.1	-.012	1.648	(.007)	641.4	1.219	84.5
	0.00001	1159.3	.067	5.833	(.264)	17.0	11.75	62.8
	0.000025	1045.8	.074	5.546	(.271)	15.5	11.17	60.5
	0.00005	653.8	.077	5.298	(.278)	14.5	10.67	59.3
Slash	0.00010	508.0	.077	5.017	(.287)	13.7	10.11	58.9
	0.00050	295.1	.065	4.207	(.240)	13.2	8.477	61.4
	0.00100	245.2	.056	3.827	(.169)	13.4	7.711	63.0
	0.00500	172.9	.042	3.015	(.080)	12.9	6.074	64.0
	0.01000	148.9	.036	2.663	(.066)	12.6	5.366	63.5
	0.02500	121.7	.024	2.154	(.047)	13.4	4.340	63.2
	0.05000	106.5	.010	1.734	(.037)	18.1	3.493	63.8

<u>Exhibit 5(c)</u>. Results on one-sample biweight - "t", n = 10, borrowing denominators from 4 samples.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	65.4	-.012	4.707	(.051)	50.9	3.043	97.0
	0.000025	68.2	-.011	4.426	(.044)	52.1	2.861	97.2
	0.00005	70.2	-.011	4.210	(.039)	53.1	2.722	97.3
Dist'n.	0.00010	72.2	-.011	3.991	(.031)	54.5	2.580	97.5
Gaussian	0.00050	77.2	-.011	3.465	(.022)	59.1	2.240	97.7
	0.00100	79.5	-.011	3.228	(.017)	62.0	2.087	97.8
	0.00500	85.3	-.011	2.644	(.012)	74.3	1.709	98.0
	0.01000	87.8	-.010	2.371	(.009)	85.6	1.533	98.0
	0.02500	91.1	-.009	1.980	(.007)	123.3	1.280	98.0
	0.05000	93.4	-.009	1.650	(.006)	277.3	1.067	98.0
	0.00001	51.2	-.018	4.637	(.060)	59.8	3.439	86.8
	0.000025	54.8	-.018	4.363	(.051)	62.1	3.235	86.6
	0.00005	57.4	-.017	4.152	(.043)	64.3	3.078	86.5
Dist'n.	0.00010	60.2	-.017	3.937	(.032)	67.3	2.919	86.3
One-Wild	0.00050	67.3	-.016	3.422	(.023)	77.8	2.537	86.0
	0.00100	70.6	-.016	3.190	(.018)	85.1	2.365	85.8
	0.00500	79.1	-.014	2.617	(.013)	121.9	1.941	85.2
	0.01000	82.5	-.014	2.349	(.011)	165.4	1.742	85.0
	0.02500	87.7	-.013	1.964	(.009)	756.0	1.456	84.6
	0.05000	91.2	-.012	1.639	(.007)	∞	1.215	84.3
	0.00001	2273.2	.101	6.107	(.277)	15.0	12.39	54.1
	0.000025	1515.3	.106	5.802	(.291)	13.7	11.77	53.3
	0.00005	1096.0	.108	5.540	(.299)	12.8	11.24	53.1
Dist'n.	0.00010	791.9	.108	5.247	(.293)	12.1	10.65	53.2
Slash	0.00050	392.9	.091	4.377	(.257)	11.5	8.881	56.7
	0.00100	303.8	.077	3.951	(.193)	11.8	8.018	59.4
	0.00500	189.9	.050	3.038	(.089)	12.4	6.163	62.4
	0.01000	152.6	.041	2.661	(.071)	12.6	5.400	62.2
	0.02500	121.4	.024	2.137	(.048)	14.6	4.336	62.6
	0.05000	105.6	.008	1.715	(.037)	22.7	3.479	63.9

Exhibit 5(d). Results on one-sample biweight - "t", n = 10, borrowing denominators from 5 samples.

Dist'n.	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	86.1	-.004	4.898	(.043)	36.6	2.223	94.7
	0.000025	86.1	-.005	4.589	(.037)	37.2	2.082	95.2
	0.00005	85.8	-.005	4.353	(.032)	37.7	1.975	95.5
Dist'n.	0.00010	86.1	-.005	4.116	(.027)	38.2	1,868	95.8
Gaussian	0.00050	87.8	-.006	3.555	(.019)	39.7	1.613	96.4
	0.00100	88.9	-.005	3.306	(.016)	40.5	1.500	96.6
	0.00500	92.1	-.005	2.696	(.011)	42.9	1.224	97.0
	0.01000	93.5	-.005	2.414	(.010)	44.5	1.095	97.1
	0.02500	95.4	-.005	2.011	(.008)	47.9	0.913	97.2
	0.05000	96.7	-.004	1.674	(.006)	52.8	0.760	97.2
	0.00001	56.1	-.017	4.764	(.051)	45.5	2.281	96.1
	0.000025	58.9	-.016	4.471	(.043)	46.0	2.140	96.1
	0.00005	61.3	-.016	4.246	(.036)	48.0	2.033	96.1
Dist'n.	0.00010	63.8	-.016	4.020	(.028)	49.5	1.924	96.1
One-Wild	0.00050	70.3	-.015	3.481	(.022)	54.3	1.666	96.0
	0.00100	73.4	-.014	3.240	(.017)	57.4	1.551	95.9
	0.00500	81.0	-.013	2.648	(.013)	70.4	1.268	95.6
	0.01000	84.4	-.012	2.373	(.011)	82.2	1.136	95.4
	0.02500	88.9	-.011	1.980	(.092)	120.9	0.948	95.1
	0.05000	92.1	-.011	1.650	(.007)	272.0	0.790	94.9
	0.00001	118.3	.005	5.006	(.144)	31.9	6.637	83.2
	0.000025	118.3	.005	4.697	(.123)	31.5	6.227	81.8
	0.00005	119.1	.006	4.464	(.113)	31.0	5.919	80.5
Dist'n.	0.00010	119.3	.006	4.228	(.106)	30.5	5.606	78.7
Slash	0.00050	115.7	.006	3.655	(.092)	29.5	4.845	75.6
	0.00100	113.7	.006	3.397	(.084)	29.1	4.504	73.8
	0.00500	109.9	.006	2.768	(.059)	27.5	3.670	69.0
	0.01000	106.7	.005	2.471	(.051)	27.4	3.276	67.4
	0.02500	102.1	.002	2.043	(.039)	29.8	2.709	66.1
	0.05000	99.6	-.001	1.689	(.031)	35.2	2.240	65.5

Exhibit 6 (a). Results on one-sample biweight - "t", n = 20, borrowing denominators from 2 samples.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	79.2	-.006	4.634	(.026)	60.1	2.108	96.5
	0.000025	80.9	-.006	4.368	(.023)	61.0	1.987	96.7
	0.00005	82.3	-.006	4.163	(.020)	61.7	1.894	96.8
Dist'n.	0.00010	83.8	-.006	3.954	(.017)	62.6	1.799	96.8
Gaussian	0.00050	87.4	-.006	3.448	(.013)	65.0	1.569	97.0
	0.00100	88.9	-.005	3.219	(.013)	66.5	1.404	97.0
	0.00500	92.5	-.005	2.647	(.086)	71.8	1.204	97.1
	0.01000	93.9	-.005	2.377	(.075)	75.9	1.081	97.2
	0.02500	95.7	-.004	1.988	(.061)	85.7	0.904	97.2
	0.05000	96.9	-.004	1.660	(.050)	101.6	0.755	97.2
	0.00001	53.6	-.016	4.529	(.030)	83.1	2.174	96.5
	0.000025	56.9	-.016	4.271	(.026)	87.1	2.050	96.4
	0.00005	59.5	-.016	4.072	(.023)	91.2	1.955	96.4
Dist'n.	0.00010	62.4	-.015	3.870	(.019)	96.2	1.857	96.3
One-Wild	0.00050	69.7	-.014	3.379	(.015)	114.4	1.622	96.0
	0.00100	73.1	-.014	3.157	(.013)	127.8	1.515	95.9
	0.00500	81.2	-.013	2.600	(.010)	210.0	1.248	95.5
	0.01000	84.6	-.012	2.337	(.009)	362.2	1.122	95.3
	0.02500	89.0	-.011	1.957	(.007)	∞	0.940	95.0
	0.05000	92.1	-.010	1.636	(.006)	∞	0.785	94.8
	0.00001	83.9	-.004	4.655	(.113)	57.3	6.246	88.8
	0.000025	92.6	-.002	4.410	(.102)	54.3	5.917	86.8
	0.00005	100.9	.0003	4.224	(.093)	51.1	5.608	84.7
Dist'n.	0.00010	111.0	.003	4.036	(.083)	47.1	5.416	81.9
Slash	0.00050	134.4	.011	3.582	(.072)	36.3	4.806	73.7
	0.00100	138.8	.014	3.367	(.072)	32.0	4.518	70.1
	0.00500	127.3	.010	2.778	(.062)	26.2	3.728	64.3
	0.01000	117.2	.013	2.476	(.055)	26.6	3.322	63.4
	0.02500	105.4	.006	2.635	(.042)	33.1	2.730	63.4
	0.05000	98.9	-.002	1.669	(.033)	32.8	2.240	64.1

Exhibit 6 (b). Results on one-sample biweight - "t", n = 20, borrowing denominators from 3 samples.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
Dist'n. Gaussian	0.00001	88.0	-.003	4.550	(.026)	77.1	2.072	95.8
	0.000025	87.6	-.004	4.294	(.021)	79.1	1.956	96.1
	0.00005	87.6	-.004	4.096	(.019)	80.9	1.866	96.3
	0.00010	88.1	-.004	3.895	(.016)	82.7	1.774	96.5
	0.00050	89.2	-.004	3.406	(.012)	87.9	1.551	96.7
	0.00100	90.9	-.004	3.184	(.010)	90.8	1.450	96.8
	0.00500	93.5	-.004	2.626	(.008)	101.5	1.196	97.0
	0.01000	94.6	-.004	2.361	(.006)	109.9	1.075	97.1
	0.02500	96.1	-.004	1.978	(.005)	131.2	0.901	97.1
	0.05000	97.2	-.004	1.654	(.004)	171.8	0.753	97.1
Dist'n. One-Wild	0.00001	53.0	-.016	4.423	(.021)	137.2	2.126	96.4
	0.000025	57.0	-.015	4.183	(.019)	145.8	2.011	96.3
	0.00005	60.1	-.015	3.997	(.017)	155.1	1.921	96.1
	0.00010	63.3	-.014	3.805	(.014)	167.3	1.829	96.0
	0.00050	71.0	-.013	3.337	(.012)	222.1	1.604	95.7
	0.00100	74.3	-.013	3.122	(.011)	278.6	1.501	95.5
	0.00500	82.0	-.012	2.580	(.009)	651.2	1.240	95.2
	0.01000	85.2	-.011	2.322	(.008)	∞	1.116	95.0
	0.02500	89.3	-.011	1.947	(.007)	∞	0.936	94.8
	0.05000	92.3	-.010	1.629	(.005)	∞	0.783	94.7
Dist'n. Slash	0.00001	94.7	-.001	4.574	(.077)	71.4	6.175	89.3
	0.000025	121.8	.004	4.375	(.077)	59.7	5.907	85.4
	0.00005	141.6	.009	4.217	(.079)	52.1	5.694	82.0
	0.00010	158.2	.013	4.049	(.081)	45.3	5.467	78.2
	0.00050	171.6	.021	3.609	(.083)	33.3	4.873	69.3
	0.00100	165.1	.023	3.389	(.084)	29.8	4.576	66.1
	0.00500	136.0	.021	2.782	(.064)	25.7	3.756	61.9
	0.01000	123.0	.017	2.479	(.056)	26.1	3.347	61.3
	0.02500	108.2	.009	2.036	(.043)	32.3	2.749	61.8
	0.05000	99.6	-.001	1.666	(.034)	73.5	2.249	63.0

<u>Exhibit 6(c)</u>. Results on one-sample biweight - "t", $n = 20$, borrowing denominators from 4 samples.

	Tail Pr.	$r_1(\alpha)$	$r_2(\alpha)$	Crit. Pt.	Stnd. Error	D.F.	ECIL	Efficiency
	0.00001	96.0	-.003	4.487	(.023)	97.6	2.045	95.9
	0.000025	89.2	-.003	4.240	(.019)	101.5	1.933	96.2
	0.00005	88.4	-.004	4.049	(.017)	104.4	1.845	96.4
Dist'n.	0.00010	88.4	-.004	3.854	(.014)	107.5	1.756	96.6
Gaussian	0.00050	88.7	-.004	3.378	(.010)	116.3	1.540	96.8
	0.00100	91.3	-.004	3.160	(.009)	121.3	1.440	96.9
	0.00500	93.7	-.004	2.612	(.007)	140.7	1.190	97.0
	0.01000	94.8	-.004	2.351	(.006)	157.0	1.071	97.1
	0.02500	96.2	-.004	1.972	(.005)	203.5	0.899	97.1
	0.05000	97.3	-.003	1.650	(.004)	323.6	0.752	92.1
	0.00001	55.3	-.014	4.372	(.016)	200.0	2.103	96.0
	0.000025	69.2	-.014	4.140	(.015)	217.0	1.992	95.8
	0.00005	62.2	-.013	3.959	(.014)	263.8	1.905	95.7
Dist'n.	0.00010	65.2	-.013	3.772	(.012)	290.8	1.815	95.6
One-Wild	0.00050	72.4	-.012	3.314	(.011)	416.9	1.594	95.4
	0.00100	75.5	-.012	3.102	(.009)	600.2	1.493	95.2
	0.00500	82.7	-.011	2.568	(.008)	∞	1.236	95.0
	0.01000	85.7	-.011	2.313	(.007)	∞	1.113	94.9
	0.02500	89.6	-.010	1.942	(.006)	∞	0.934	94.7
	0.05000	92.4	-.010	1.625	(.005)	∞	0.782	94.6
	0.00001	251.7	.021	4.743	(.108)	47.4	6.434	80.4
	0.000025	261.1	.024	4.520	(.112)	42.2	6.131	77.2
	0.00005	262.8	.027	4.343	(.107)	38.4	5.892	74.3
Dist'n.	0.00010	258.7	.029	4.159	(.105)	34.8	5.642	71.2
Slash	0.00050	225.4	.034	3.684	(.094)	27.5	4.997	64.1
	0.00100	203.8	.034	3.451	(.095)	25.0	4.682	61.7
	0.00500	151.4	.030	2.821	(.068)	21.9	3.827	58.8
	0.01000	131.9	.024	2.505	(.060)	22.4	3.399	58.8
	0.02500	111.4	.012	2.046	(.044)	28.8	2.776	60.0
	0.05000	100.3	.001	1.665	(.035)	77.1	2.258	62.0

Exhibit 6 (d). Results on one-sample biweight - "t", n = 20, borrowing denominators from 5 samples.

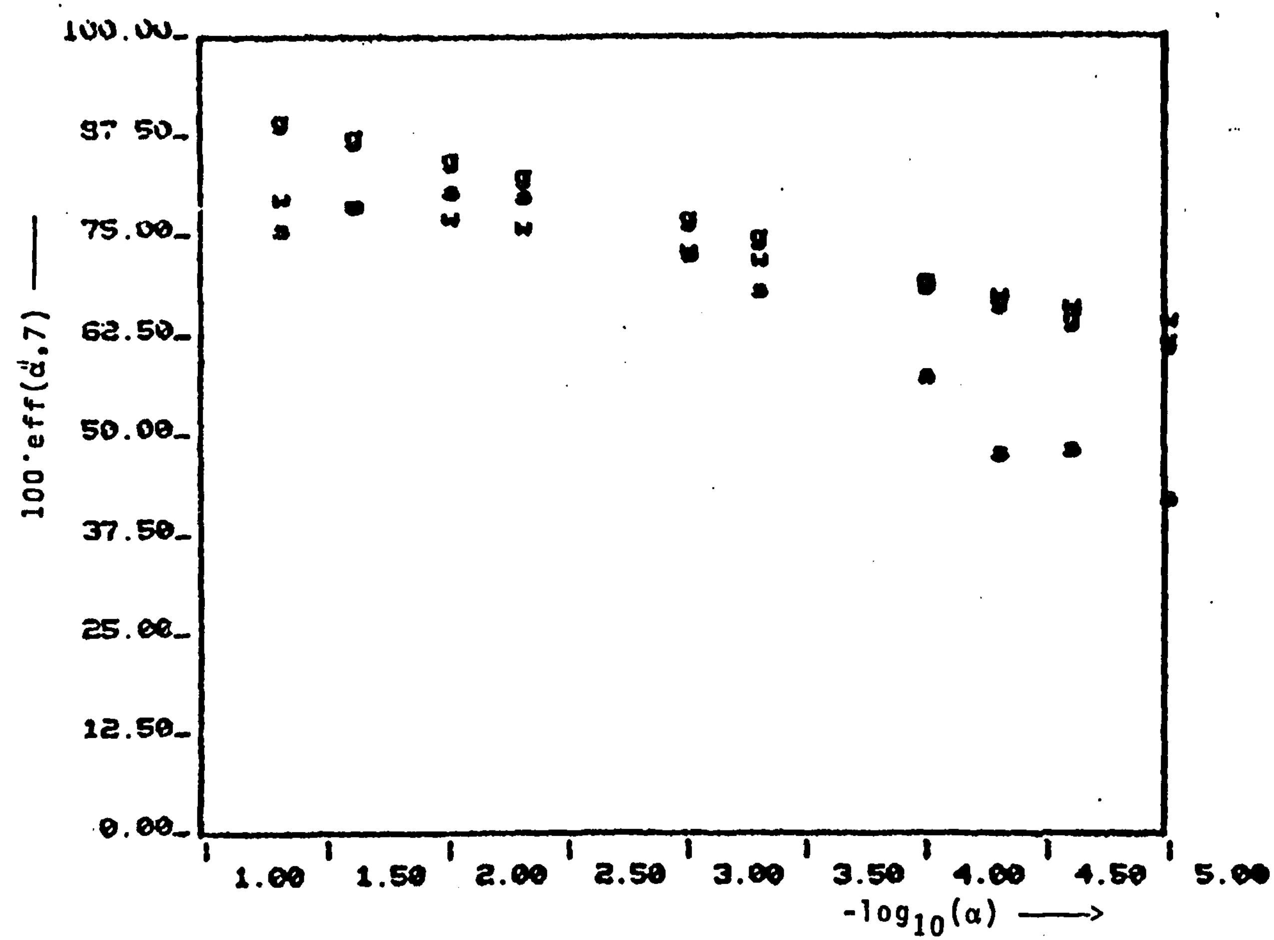

Exhibit 7(a). Approximate ECIL efficiency of biweight-"t", n=10, one-sample denominator (approximated by Student's-t on 7 d.f.)

g=Gaussian; w=One-Wild; s=Slash

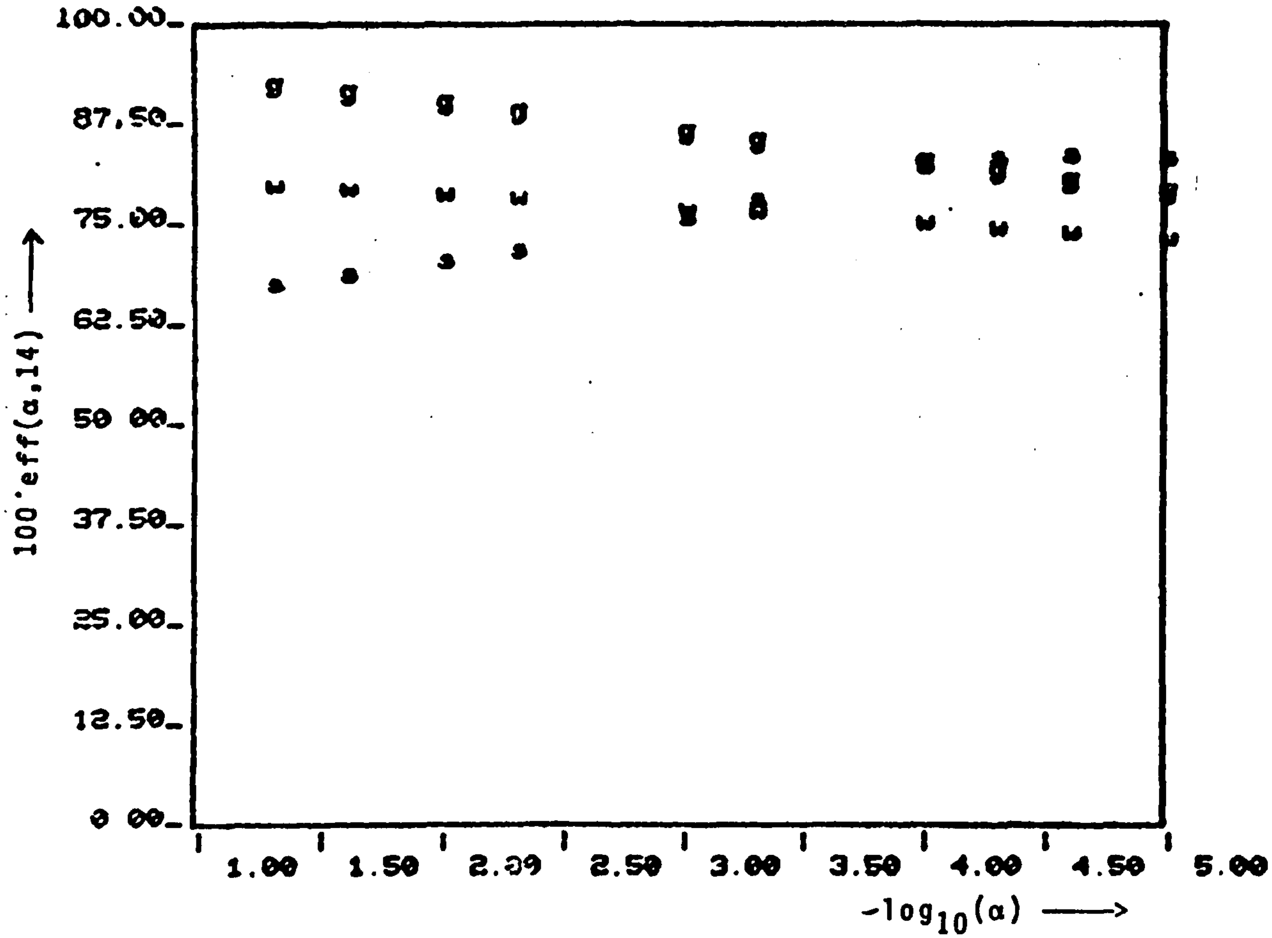

Exhibit 7 (b): Approximate ECIL efficiency of biweight-"t", n=10, 2-sample denominator (approximated by Student's-t_{14}).
g=Gaussian; w=One-wild; s=Slash

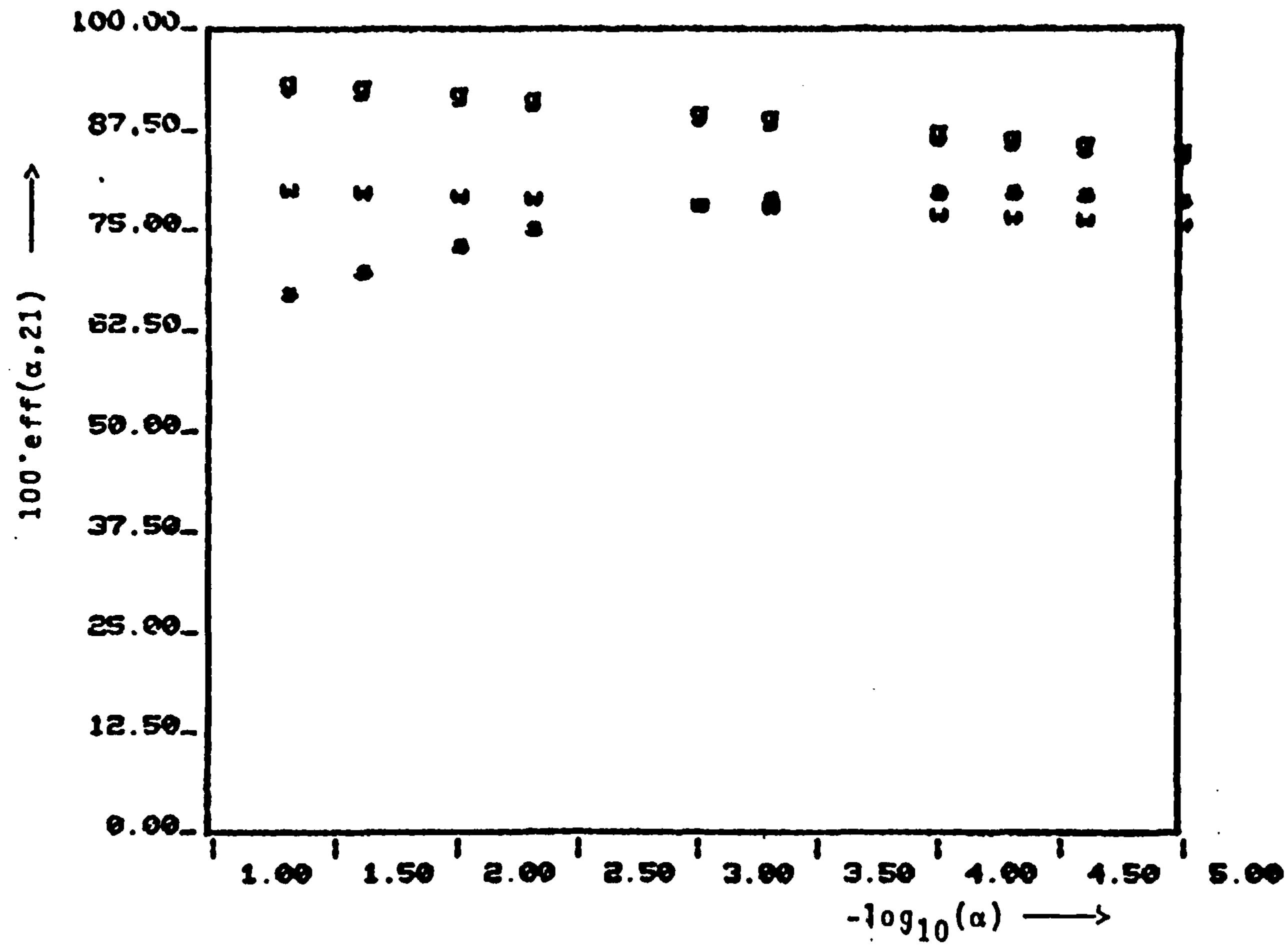

Exhibit 7 (c): Approximate ECIL efficiency of biweight-"t", n=10, 3-sample denominator (approximated by Student's-t on 21 d.f.)

g=Gaussian; w=One-Wild; s=Slash

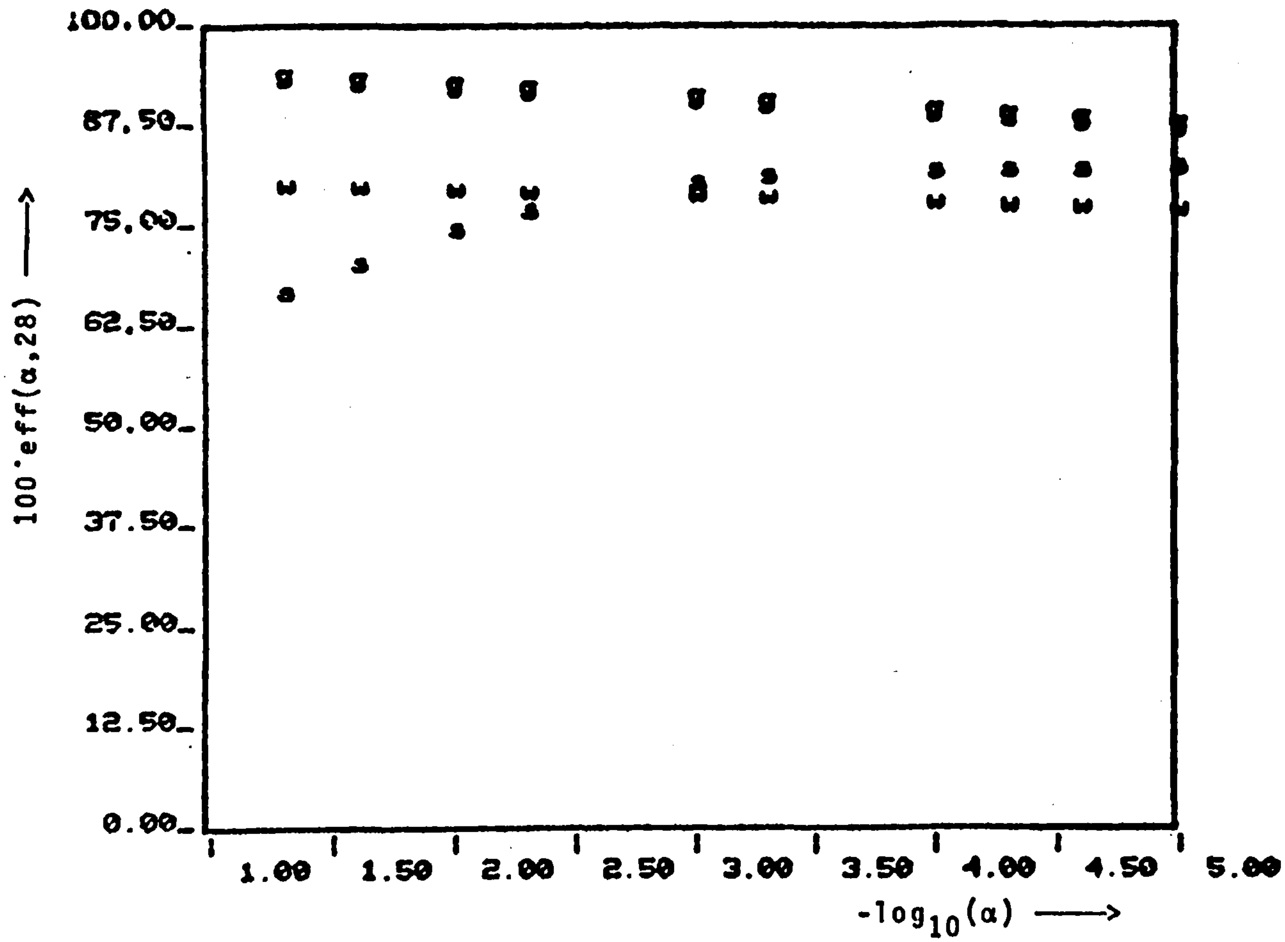

Exhibit 7 (d): Approximate ECIL efficiency of biweight-"t", n=10,
4-sample denominator (approximated by Student's-t on 28 d.f.)

g=Gaussian; w=One-Wild; s=Slash

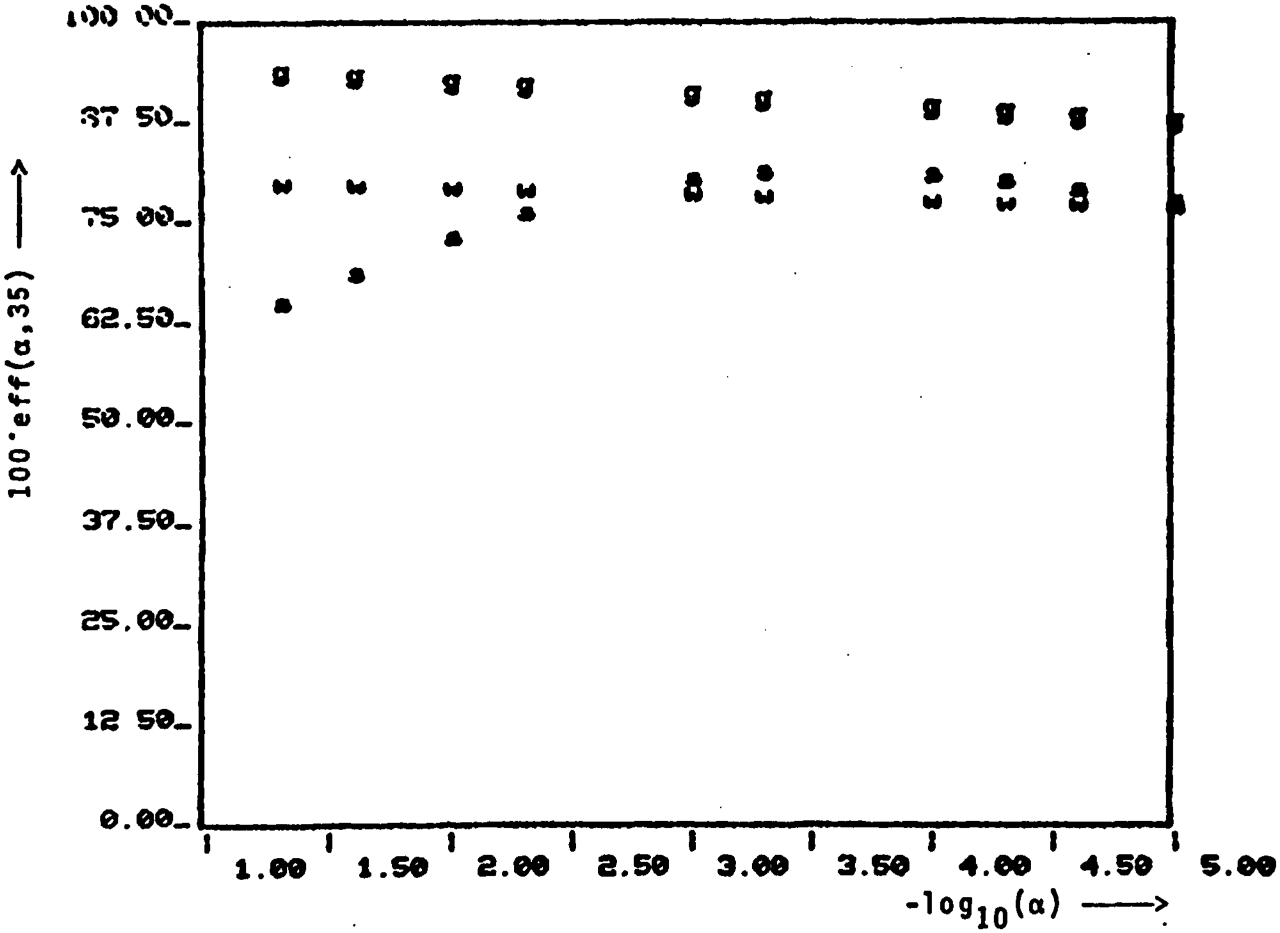

Exhibit 7 (e): Approximate ECIL efficiency of biweight-"t", n=10, 5-sample denominator (approximated by Student's-t on 35 d.f.)

g=Gaussian; w=One-Wild; s=Slash

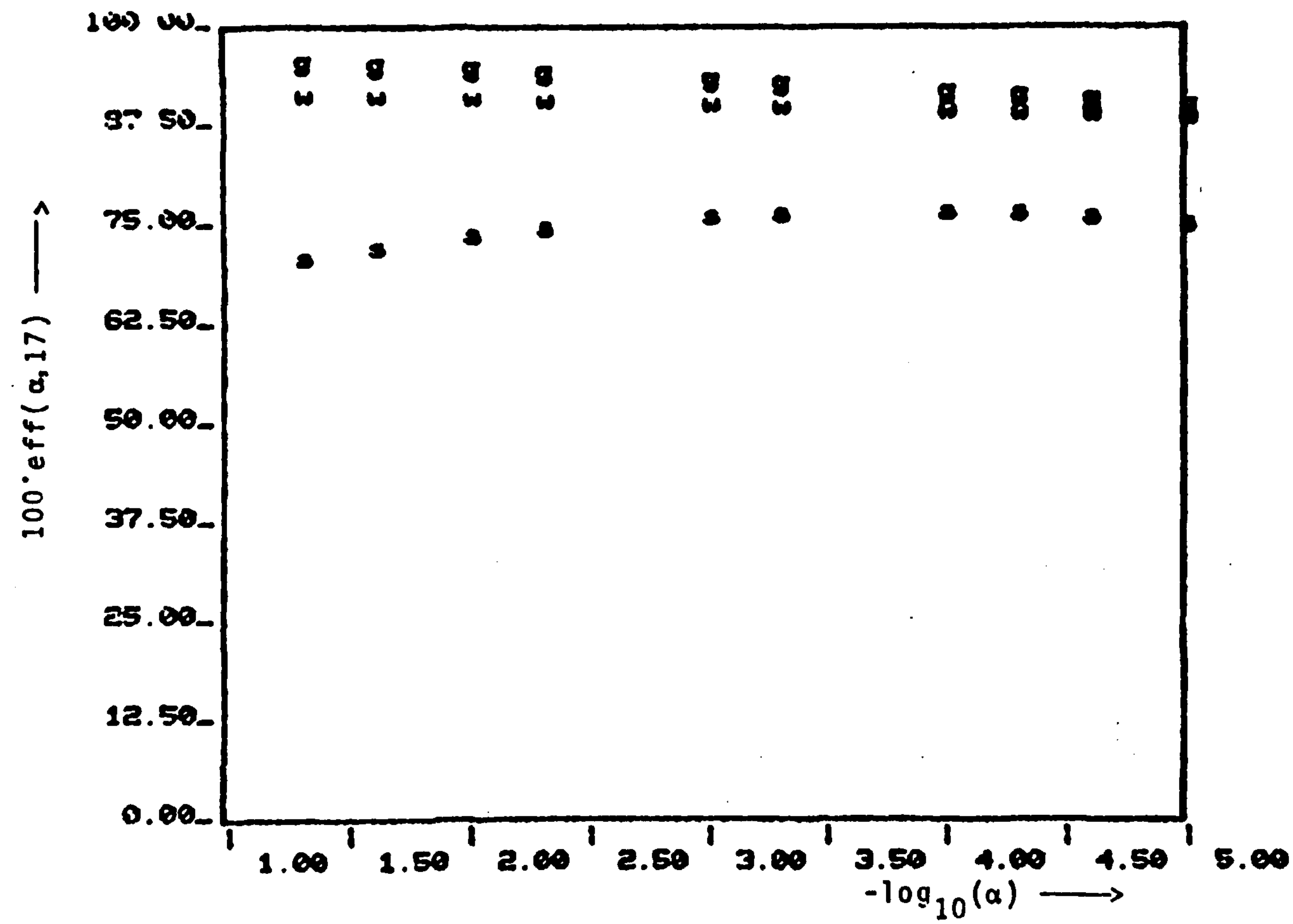

Exhibit 8(a): Approximate ECIL efficiency of biweight-"t", n=20, 1-sampledenominator (approximated by Student's-t on 17 d.f.)

g=Gaussian; w=One-Wild; s=Slash

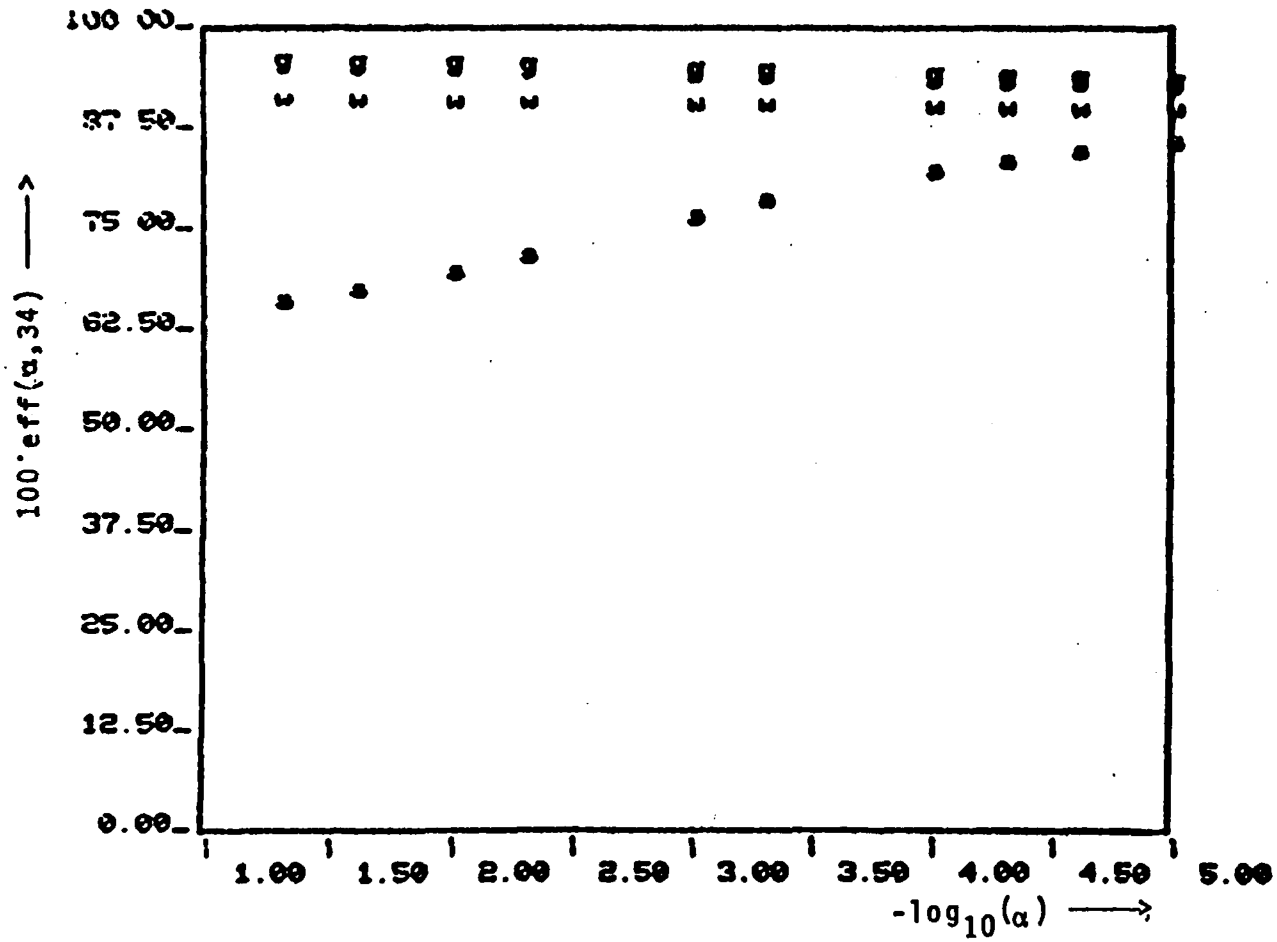

Exhibit 8(b): Approximate FCIL efficiency of biweight-"t", n=20 2-sample denominator (approximated by Student's-t on 34 d.f.)

g=Gaussian; w=One-Wild; s=Slash

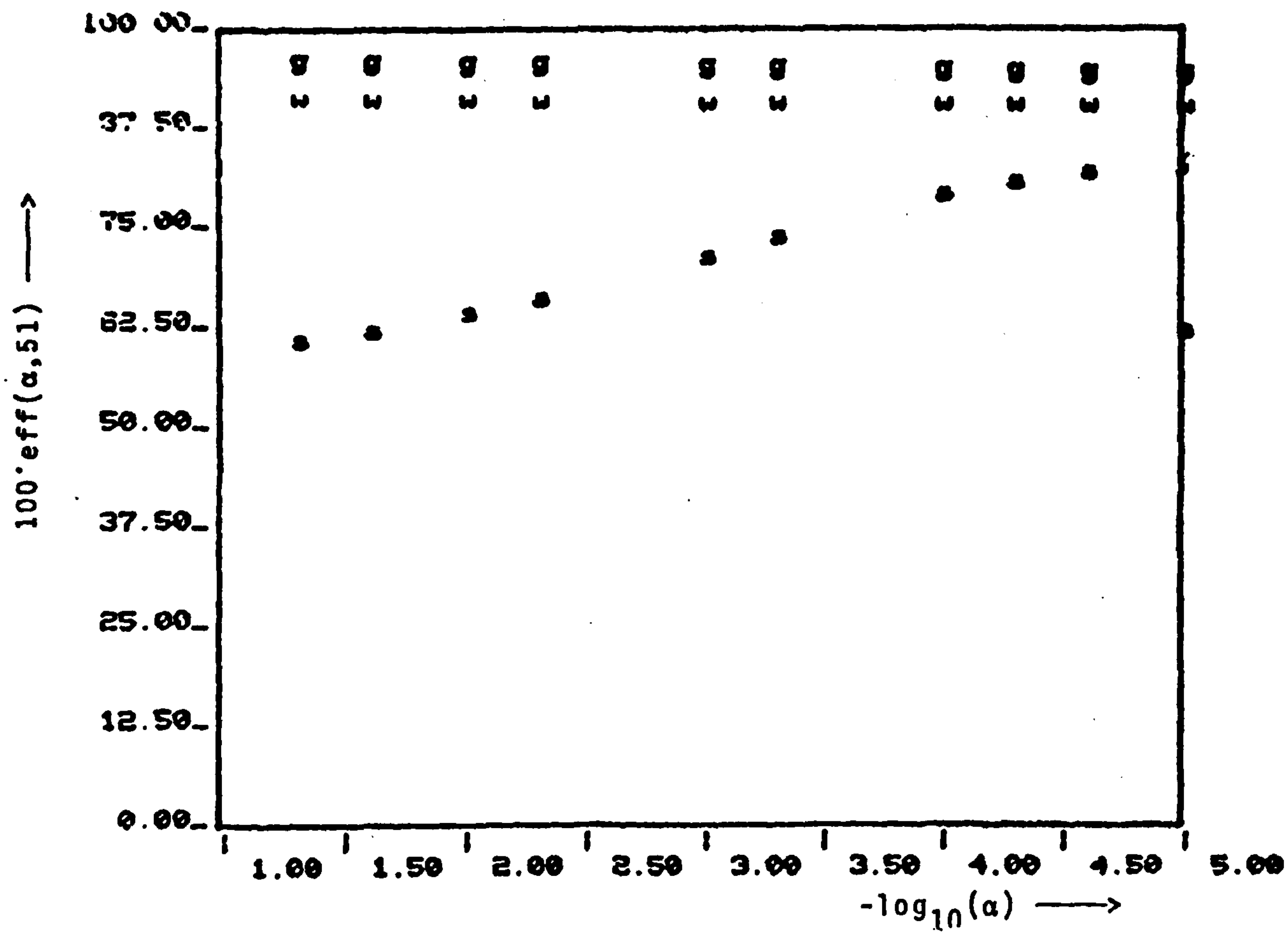

Exhibit 8 (c): Approximate ECIL efficiency of biweight-"t", n=20, 3-sample denominator (approximated by Student's-t on 51 d.f.)

g=Gaussian; w=One-Wild; s=Slash

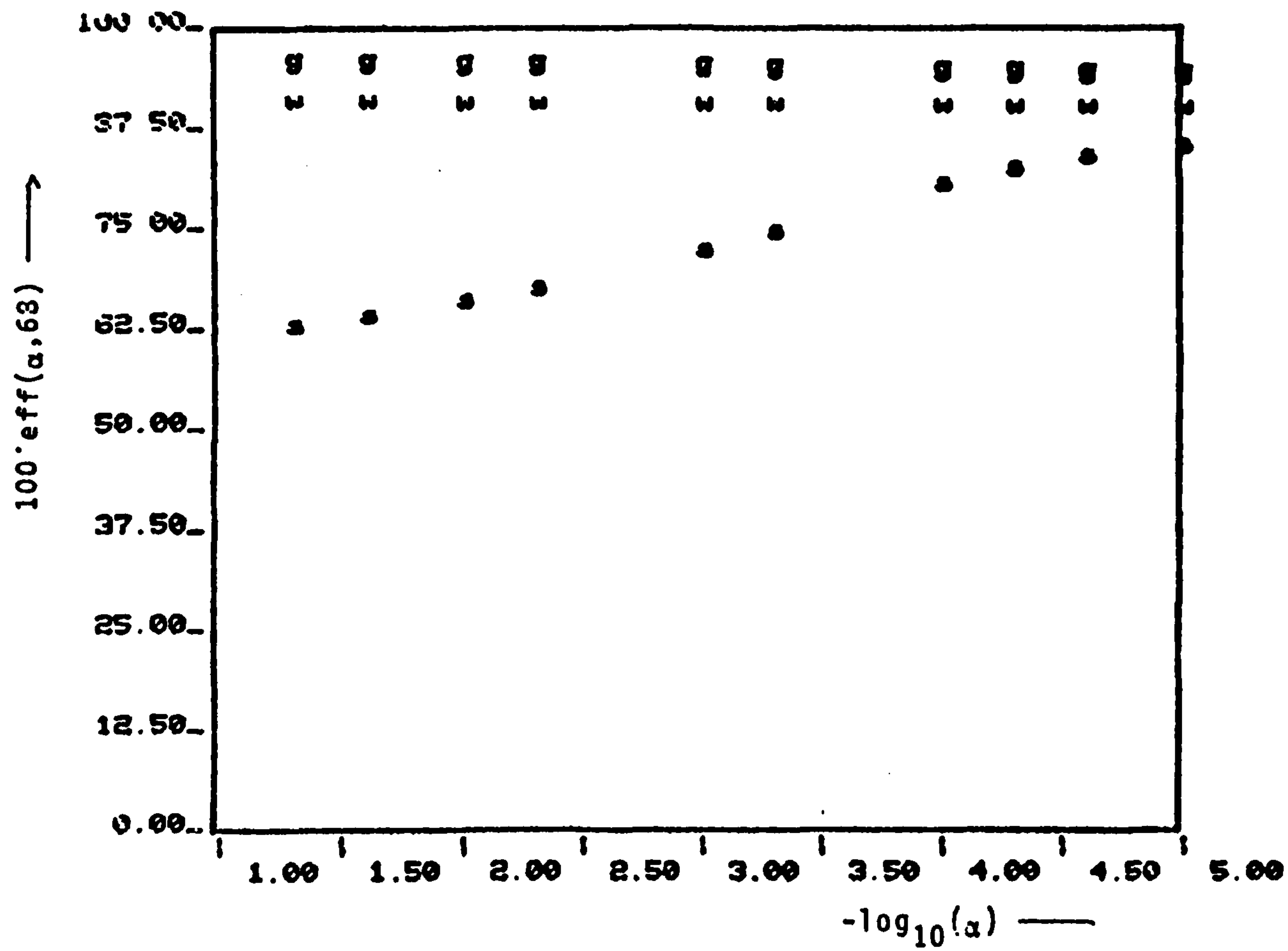

Exhibit 8(d): Approximate ECIL efficiency of biweight-"t", n=20, 4-sample denominator (approximated by Student's-t on 68 d.f.)

g=Gaussian; w=One-Wild; s=Slash

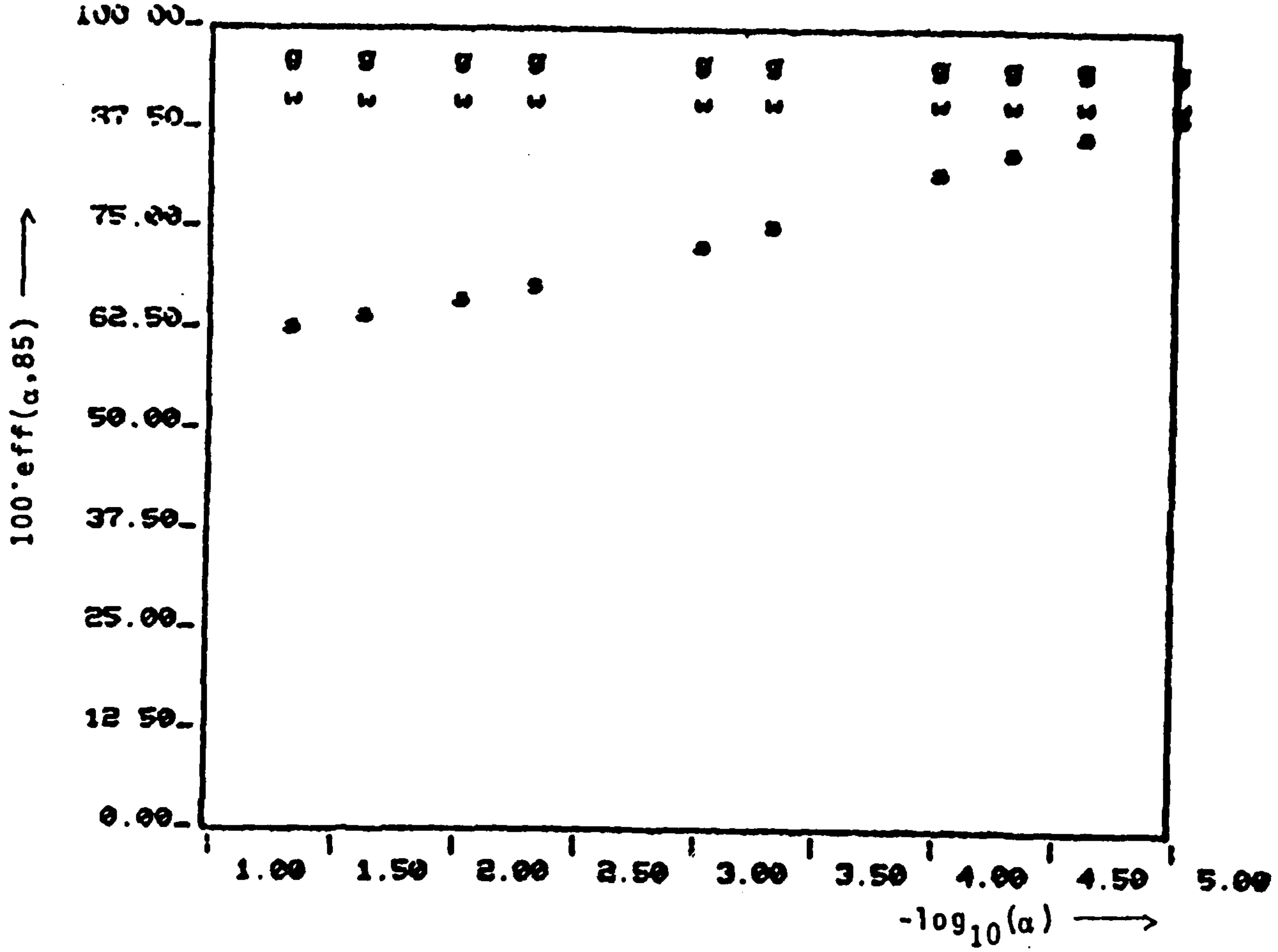

Exhibit 8 (e): Approximate ECIL efficiency of biweight-"t", n=20, 5-sample denominator (approximated by Student's-t on 85 d.f.)
g=Gaussian; w=One-Wild; s=Slash

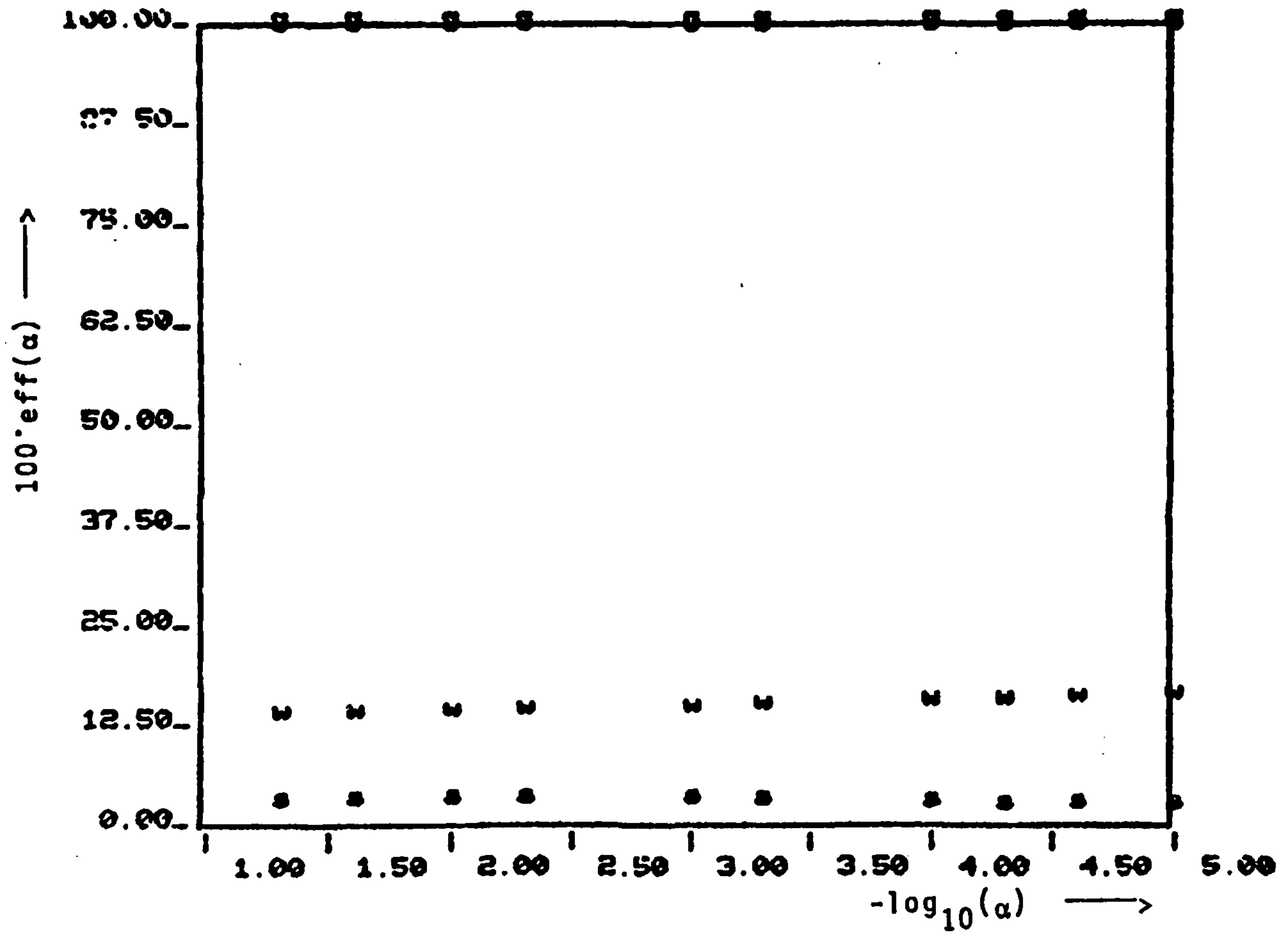

Exhibit 9: ECIL efficiency of classical Student's-t (9 d.f.) for 3 situations (Gaussian, One-Wild, Slash), n=10.

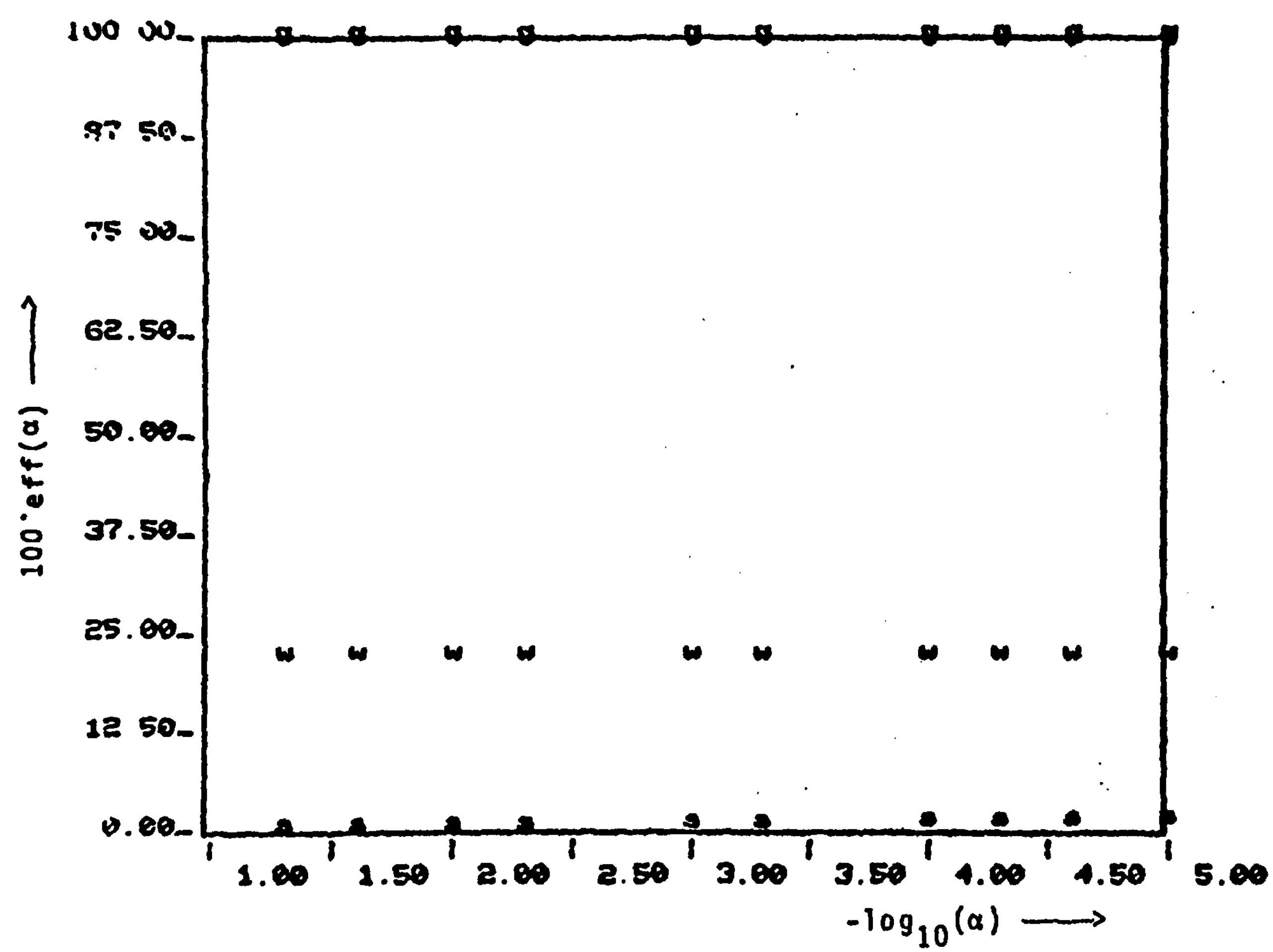

Exhibit 10: ECIL efficiency of classical Student's-t (19 d.f.) for 3 situations (Gaussian, One-Wild, Slash),,n=20.

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